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ABSTRACT

It is assumed that the deformations of a thin cylindrical elastic shell, subjected to a uniform lateral pressure, are adequately described by the Donnell shell equations. The surface of the shell is perforated by a rectangular cutout. The tangential membrane displacements and the normal membrane stresses vanish on the simply supported edges of the hole and the ends of the cylinder. Numerical solutions are obtained by replacing the boundary value problem by a finite difference approximation and solving the resulting system of linear algebraic equations by an iteration procedure.

1. Introduction.

In this paper we study the deformations of perforated cylindrical elastic shells that are subjected to surface and boundary forces. The presence of a hole in the shell's wall significantly alters the distribution and magnitudes of the stresses. Approximate procedures, which are restricted in their applicability, have been proposed by various authors [1-6] to determine these stresses. The paper by Savin [4] also contains a general discussion of shell cutout problems and references to previous work.

For a specific cutout problem we propose a numerical method to determine the stresses and displacements in the shell. However, with suitable modifications the method can be extended to include a variety of cut-out shapes and other boundary and loading conditions.

For the specific problem investigated the perforation is assumed to be rectangular, i.e. the edges of the cutout coincide with generators and parallel circles of the cylinder's mid-surface. The shell is loaded by a uniform pressure applied normal to the surface. On the simply supported boundaries the tangential membrane displacements and the normal membrane stresses vanish. It is assumed that the deformations are adequately described by the Donnell thin shell theory, see e.g. [7,8].

The method consists of replacing the boundary value problem of the Donnell theory by a finite difference approximation. The resulting algebraic equations are then solved by a special iteration method involving two acceleration parameters. This procedure is described in Sec. 3. The computational methods required to apply the method are outlined in Sec. 4. In Sec. 5 the results of the computation are discussed and graphs of stresses and displacements are shown.

The numerical results indicate that the stresses are large near the corner of the rectangular cutout. These high stresses may have slowed the convergence of the iterative procedure. We therefore conjecture that for problems with smoother cutouts the method may be more efficient.

2. Formulation.

The cylindrical shell is of mean radius R, thickness t and length 2L. A cylindrical coordinate system r, θ, z is employed where r and θ are plane polar coordinates and z is the axial distance measured from one end of the cylinder.

Thus the coordinates are in the ranges $R - t/2 \leq r \leq R + t/2$, $-\pi \leq \theta \leq \pi$, $-L \leq z \leq L$. The dimensionless coordinates x and y are defined by the relations*

$$
(2.1) \qquad \begin{aligned} x &\equiv z/L \,, & |x| &\leq 1 \,, \\ y &= (R/L)\theta \,, & |y| &\leq (\pi R/L) \equiv \ell \,, \end{aligned}
$$

The wall of the shell is perforated by a rectangular hole whose edges are on the lines $x = \pm\, \alpha$ and $y = \pm\, \beta\ell$ where $0 \leq \alpha$, $\beta < 1$. The shell is deformed by a uniform pressure p applied normal to the surface. The ends of the shell and the edges of the hole are simply supported and the normal membrane stresses and the tangential membrane displacements vanish there. The solution is therefore assumed to be symmetric with respect to the planes $x = 0$ and $y = 0$. Thus we consider the region D, see Fig. 1, which is one quarter of the cylindrical surface. Symmetry conditions must be specified along the following three boundary lines of D: $\alpha \leq x \leq 1$, $y = 0$; $0 \leq x \leq 1$, $y = \ell$; $x = 0$, $\beta\ell \leq y \leq \ell$.

The Donnell theory for cylindrical shells, see e.g. [7,8], can be formulated as a boundary value problem for two coupled fourth order partial differential equations in the two dependent variables $W(z,\theta)$ and $F(z,\theta)$. Here, W is the normal displacement of the midsurface $r = R$ of the cylinder and EF is the membrane

*The coordinates are scaled so that the cylinder is of dimensionless length equal to 2. However, there are other scalings, say with L in (2.1) replaced by any characteristic dimension of the shell, which yield formulations similar to the one given below.

stress function where E is Young's modulus. In dimensionless variables the differential equations of the Donnell theory are

(2.2a) $$\triangle^2 w + \tau\frac{\partial^2 f}{\partial x^2} = 1, \quad \triangle^2 f = \tau\frac{\partial^2 w}{\partial x^2}$$

where $\triangle^2$ is the biharmonic operator,

(2.2b) $$\triangle^2 \equiv \frac{\partial^4}{\partial x^4} + 2\,\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \quad .$$

On the boundary of D we specify that

(2.3a)
$$w = \frac{\partial^2 w}{\partial x^2} = f = \frac{\partial^2 f}{\partial x^2} = 0 \ , \quad \text{for } x = 1 \ , \quad 0 \leq y \leq \ell,$$

$$w = \frac{\partial^2 w}{\partial x^2} = f = \frac{\partial^2 f}{\partial x^2} = 0 \ , \quad \text{for } x = \alpha \ , \quad 0 \leq y \leq \beta\ell,$$

$$w = \frac{\partial^2 w}{\partial y^2} = f = \frac{\partial^2 f}{\partial y^2} = 0 \ , \quad \text{for } 0 \leq x \leq \alpha \ , \quad y = \beta\ell,$$

(2.3b)
$$\frac{\partial w}{\partial y} = \frac{\partial^3 w}{\partial y^3} = \frac{\partial f}{\partial y} = \frac{\partial^3 f}{\partial y^3} = 0 \ , \quad \text{for } y = 0 \ , \quad \alpha \leq x \leq 1 \ , \quad \text{and} \quad y = \ell \ , \quad 0 \leq x \leq 1 \ ,$$

$$\frac{\partial w}{\partial x} = \frac{\partial^3 w}{\partial x^3} = \frac{\partial f}{\partial x} = \frac{\partial^3 f}{\partial x^3} = 0 \ , \quad \text{for } x = 0 \ , \quad \beta\ell \leq y \leq \ell .$$

The dimensionless variables in (2.2) and (2.3) are defined by

$$w(x,y) \equiv CW(z,\theta) \quad , \quad f(x,y) \equiv [12(1-\nu^2)]^{1/2} \; C \; \frac{F(z,\theta)}{t}$$

(2.4)

$$C \equiv \frac{E}{12(1-\nu^2)pL} \left(\frac{t}{L}\right)^3 \quad , \quad \tau \equiv [12(1-\nu^2)]^{1/2} \; \frac{L^2}{Rt} \quad ,$$

where ν is Poisson's ratio. The membrane stresses σ_z^o, σ_θ^o and $\sigma_{z\theta}^o$ and the corresponding dimensionless stresses Σ_x^o , Σ_y^o and Σ_{xy}^o are given in terms of the dimensionless stress function $f(x,y)$ by

$$\Sigma_x^o(x,y) \equiv A\sigma_z^o(z,\theta) = \frac{\partial^2 f(x,y)}{\partial y^2} \; , \; \Sigma_y^o(x,y) = A\sigma_\theta^o(z,\theta) = \frac{\partial^2 f(x,y)}{\partial x^2} \; ,$$

(2.5)

$$\Sigma_{xy}^o(x,y) \equiv A\sigma_{z\theta}^o(z,\theta) = - \frac{\partial^2 f(x,y)}{\partial x \partial y}$$

where

$$A^{-1} \equiv [12(1-\nu^2)]^{1/2} p(L/t)^2 \; .$$

The quantities σ_z', σ_θ' and $\sigma_{z\theta}'$, which are proportional to the bending moments, and the dimensionless bending stresses m_x, m_y and m_{xy} are given in terms of $w(x,y)$ by,

$$m_x(x,y) \equiv B\sigma_z'(z,\theta) = \frac{\partial^2 w(x,y)}{\partial x^2} + \nu \frac{\partial^2 w(x,y)}{\partial y^2} \quad ,$$

(2.6)

$$m_y(x,y) \equiv B\sigma_\theta'(z,\theta) = \frac{\partial^2 w(x,y)}{\partial y^2} + \nu \frac{\partial^2 w(x,y)}{\partial x^2} \quad ,$$

$$m_{xy}(x,y) \equiv (1-\nu)B\sigma_{z\theta}'(z,\theta) = \frac{\partial^2 w(x,y)}{\partial x \partial y} \quad ,$$

where

$$B \equiv - \frac{t}{12p}\left(\frac{t}{L}\right)^2 \quad .$$

If the displacements of the midsurface of the shell in the z and θ directions are $U(z,\theta)$ and $V(z,\theta)$ respectively, then the dimensionless displacements $u(x,y)$ and $v(x,y)$ are defined by

$$u(x,y) \equiv \left(\frac{R}{L}\right) \frac{U(z,\theta)}{C} \quad , \quad v(x,y) \equiv \left(\frac{R}{L}\right) \frac{V(z,\theta)}{C} \quad .$$

The membrane stress displacement relations of the Donnell theory thus reduce to [7,8],

$$\Sigma_x^o = \frac{\tau}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\left(\frac{\partial v}{\partial y} + w\right)\right], \quad \Sigma_y^o = \frac{\tau}{1-\nu^2}\left[\frac{\partial v}{\partial y} + w + \nu\frac{\partial u}{\partial x}\right] \quad ,$$

(2.7)

$$\Sigma_{xy}^o = \frac{\tau}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad .$$

The relations (2.5), (2.6) and (2.7) and the conditions (2.3) imply that the required boundary and symmetry conditions* are satisfied.

The complete formulation of the boundary value problem consists of the differential equations (2.2) and the boundary conditions (2.3). The solutions depend only on the parameters α, β, ℓ and τ. We obtain approximate solutions for fixed α, β, and ℓ and a sequence of τ values.

---

*The relations actually imply that the membrane displacements tangential to each boundary are constants. However, these constants are set equal to zero.

3. Numerical Methods.

To apply the approximate method the region D is covered with a rectilinear mesh. The mesh is of uniform spacing δ in the x and y directions. The mesh is chosen so that the boundary lines of D coincide with mesh lines. There are M + 1 mesh lines in the x direction and N + 1 in the y direction so that,

$$(3.1) \qquad \delta = 1/M = \ell/N .$$

The mesh points (x_i, y_j) are defined by,

$$(3.2) \quad x_i = i\delta , \quad y_j = j\delta , \quad i = 0,1,\ldots,M; \quad j = 0,1,\ldots N,$$

and the integers I and J are defined by,

$$x_I = \alpha , \quad y_J = \beta \ell .$$

The mesh region D_δ is defined as the set of points (x_i, y_j) interior to and on the boundary of D. The mesh region D'_δ is obtained from D_δ by excluding the mesh points: (x_M, y_j), $j = 0,\ldots,N$; (x_I, y_j), $j = 0,\ldots, J$; (x_i, y_J), $i = 0,\ldots, I$.

At each point of D'_δ the solutions $w(x_i, y_j)$, $f(x_i, y_j)$ of (2.2), (2.3) are assumed to be approximated by w_{ij}, f_{ij} which satisfy the difference equations,

$$(3.3) \quad \Delta^2_\delta f_{ij} + \tau L_x w_{ij} = 0 , \quad \Delta^2_\delta w_{ij} + \tau L_x f_{ij} = 1 , \quad (x_i, y_j) \in D'_\delta$$

Here the difference operator Δ^2_δ, obtained by replacing derivatives in the differential operator (2.2b) by centered difference quotients, is given by

$$\Delta_\delta^2 = L_x^2 + 2L_xL_y + L_y^2 . \tag{3.4a}$$

L_x and L_y are the "one-dimensional" centered second difference operators defined, for any mesh function* $\{g_{ij}\}$, by

(3.4b)

$$L_x g_{ij} = (g_{i+1,j} - 2g_{i,j} + g_{i-1,j})(\delta)^{-2} ,$$

$$L_y g_{ij} = (g_{i,j+1} - 2g_{i,j} + g_{i,j-1})(\delta)^{-2} .$$

The approximating quantities $\{w_{ij}\}$ and $\{f_{ij}\}$ must satisfy difference equivalents of the boundary conditions (2.3) where all derivatives are replaced by centered difference approximations. Fictitious mesh points exterior to D_δ are introduced to apply the difference equations.

Thus a system of $2[M(N+1) - (I+1)(J+1)]$ linear algebraic oquations in the $2[M(N+1) - (I+1)(J+1)]$ unknowns $\{w_{ij}\}$ and $\{f_{ij}\}$ is obtained. We assume that as $\delta \to 0$ the solution of the algebraic system converges to the solution of (2.2), (2.3). Hence for a sufficiently small δ the solution of the algebraic problem should yield a'close' approximation to the solution of the boundary value problem.

For large M and N, it is impractical to solve the algebraic system by a direct method, e.g. Gaussian elimination. Therefore, an iteration procedure is used to obtain approximate solutions of this system. With $w_{ij}^{(o)}$ as an initial guess, for fixed values of

*A function $\{g_{ij}\}$ defined only at points of the mesh is called a mesh function.

α, β, ℓ and τ, a sequence of iterates $[w_{ij}^{(n)}, \ f_{ij}^{(n)}]$ are defined by the recursions

$$(3.5)\quad \left.\begin{aligned} &\Delta_\delta^2 f_{ij}^{(n)} = \tau L_x w_{ij}^{(n)}, \quad \Delta_\delta^2 \bar{w}_{ij}^{(n+1)} = -\tau L_x f_{ij}^{(n)} + 1, \\ &w_{ij}^{(n+1)} = \lambda \bar{w}_{ij}^{(n+1)} + (1-\lambda) w_{ij}^{(n)} \end{aligned}\right\} \quad \begin{aligned} &n = 0,1,\ldots, \\ &(x_i, y_j) \in D_\delta' . \end{aligned}$$

Each iterate must also satisfy the difference equivalents of (2.3). The acceleration parameter λ should be determined so that the iterations (3.5) converge as rapidly as possible.

Therefore, for each n the solutions of two algebraic systems of the form

$$(3.6)\qquad \Delta_\delta^2 g_{ij}^{(n+1)} = \phi_{ij}^{(n)} \qquad\qquad (x_i, y_j) \in D_\delta'$$

are required where the mesh function $\{\phi_{ij}^{(n)}\}$ is either $\{\tau L_x w_{ij}^{(n)}\}$ or $\{-\tau L_x f_{ij}^{(n)} + 1\}$ depending upon whether $\{g_{ij}^{(n)}\}$ is $\{f_{ij}^{(n)}\}$ or $\{w_{ij}^{(n+1)}\}$. A line relaxation procedure, which is a simplification of the alternating direction method [9,10] is used to obtain a crude approximation to the solution of (3.6). A single horizontal and a single vertical sweep of the mesh is performed instead of alternating back and forth between horizontal and vertical sweeps.* Thus the approximate solution $\{g_{ij}^{(n+1)}\}$ of (3.6) is defined as the solution of the one dimensional difference equations [10]:

*A limited number of numerical experiments indicate that the convergence rate of (3.5) essentially does not change if several sweeps are made through the mesh.

$$
(3.7)\qquad \left.\begin{aligned}
(1+\rho L_y^2)g_{ij}^{(n+1)} &= g_{ij}^{(n+1/2)} + \rho L_y^2 g_{ij}^{(n)} ,\\
(1+\rho L_x^2)g_{ij}^{(n+1/2)} &= g_{ij}^{(n)} - 2\rho L_y L_x g_{ij}^{(n)} - \rho L_y^2 g_{ij}^{(n)} + \rho\phi_{ij}^{(n)}
\end{aligned}\right\} (x_i, y_j) \in D'_\delta ,
$$

where $\left\{g_{ij}^{(n+1/2)}\right\}$ and $\left\{g_{ij}^{(n+1)}\right\}$ must also satisfy the appropriate difference boundary conditions. The matrices corresponding to the one dimensional difference operators in (3.7) are easily inverted by factoring. The number ρ is a second acceleration parameter and should be determined so that the iterations (3.5) converge as rapidly as possible.

4. Computational Procedures.

Application of the numerical procedure requires that the mesh spacing be 'small'. An acceptable value of δ is determined by comparing the solutions of a sequence of test calculations with successively finer meshes. From these tests we conclude that the $\delta = 1/24$ mesh yields sufficiently accurate solutions for the range of parameters considered. The numerical criterion for convergence is that the iterations satisfy the condition,

$$
(4.1)\qquad z^{(n)} \equiv \max_{(x_i, y_j) \in D'_\delta} \left\{ |w_{ij}^{(n)} - w_{ij}^{(n-1)}| , |f_{ij}^{(n)} - f_{ij}^{(n-1)}| \right\} < 10^{-k}
$$

The value of k that is employed depends on the magnitudes of the solutions. For all our calculations we used k = 8 or 9. The condition, $\lim_{n \to \infty} z^{(n)} = 0$ is only necessary for convergence.

It is found, starting from a given initial iterate, that the number of iterations required to satisfy (4.1) depends on the values of ρ and λ. For fixed α, β, ℓ and τ, the optimal values λ_0 and ρ_0 are defined as those values which minimize the number of iterations necessary to satisfy (4.1). To obtain estimates for them a series of test calculations are employed. Thus for a fixed configuration, i.e. fixed α, β and ℓ, and usually employing a coarse mesh estimates of λ_0 and ρ_0 are determined for an increasing sequence of τ values. These values are used for the finer meshes and λ_0 and ρ_0 for other values of τ are estimated by interpolating or extrapolating. In Table I representative values are given of the acceleration parameters that were employed.

TABLE I

Acceleration Parameters Employed in the Computation

	SHELL I		SHELL II	
τ	λ	ρ	λ	ρ
340	1.0	24.6	1.0	10.0
680	.675	9.7	.55	10.0
1020	.30	9.7	.25	9.6
1360	.20	9.7	-	-
1700	.12	9.7	.10	9.6

Numerical solutions are obtained for two configurations, which we call Shells I and II, and a sequence of values of

$\tau = 340, 680, 1020, 1360, 1700$. Shells I and II are defined respectively by, $\alpha = \frac{10}{24}$, $\beta = \frac{10}{22}$, $\ell = \frac{22}{24}$ and $\alpha = 7/24$, $\beta = 7/22$, $\ell = 22/24$. The initial iterate for a given τ is taken as the 'converged' solution for the nearest τ for the same shell. When the iterations converge the dimensionless stresses are computed employing obvious finite difference equivalents of (2.5) and (2.6). It is observed that the number of iterations necessary for convergence depends on the values of the parameters α, β, ℓ and τ, For example, with a fixed configuration the number increases rapidly as τ increases, and with fixed ℓ and τ the number increases as α and β decrease. In all cases a relatively large number of iterations is needed to satisfy (4.1). For example using Shell I and $\tau = 340$, 492 iterations are needed, and for $\tau = 1020$, 2481 iterations are required. However, the computing time per iteration is relatively small, e.g. the above computations required approximately 13 and 65 minutes* respectively.

5. Presentation of Results.

Representative graphs of the dimensionless stresses and normal displacement obtained from the numerical solution for Shell I with $\tau = 1700$ are given in Fig's. 2-8. In these figures the variations with the axial distance are shown for a sequence of y values. These results indicate that near the corner of the

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*All computations employed $\nu = .3$ and were performed on the IBM-7090 computer at the A.E.C. Computing and Applied Mathematics Center of the Courant Institute of Mathematical Sciences, New York University.

cutout, $x = \alpha$, $y = \beta\ell$, the numerical stresses are quite large. The hole apparently affects the stress distribution everywhere in the shell. The hoop membrane stress Σ_y^0 and the axial bending stress Σ_x' are influenced more by the hole than the other stresses. If $\alpha = \beta = 0$, the stresses and displacements obtained from the solution of the boundary value problem are independent of y. These quantities are indicated with a super-bar and are given by,

$$\bar{w}(x) = \frac{1}{\tau^2} - \frac{(\cos \beta x \cosh \beta x + \tan \beta \tanh \beta \sin \beta x \sinh \beta x)}{\tau^2 \cos \beta \cosh \beta (1+\tan^2\beta \tanh^2\beta)} ,$$

$$(5.1) \quad \bar{\Sigma}_x^o(x) = \bar{\Sigma}_{xy}^o(x) = \bar{\Sigma}_{xy}'(x) \equiv 0 , \quad \bar{\Sigma}_y^o(x) = \tau \bar{w}(x) ,$$

$$\bar{\Sigma}_x'(x) = \frac{1}{\nu} \bar{\Sigma}_y'(x) = \frac{d^2\bar{w}}{dx^2}$$

where

$$\tau^2 = 4\beta^4 .$$

These results are shown by the dotted curves in Fig's. 3, 5, 6 and 8. In Fig. 8, the lateral displacement $\bar{w}(x)$ coincides, within the resolution of the graph, with the numerical results for the line $y = 0$. These figures indicate that the cutout and its corner may increase the magnitudes of the stresses and displacements by large factors.

Although their magnitudes differ, the form of the stresses and displacements for other values of τ and for Shell II are similar to those shown in Fig's. 2-8. A simple asymptotic solution is obtained for large τ by dividing the differential equations by τ and letting $\tau \rightarrow \infty$ while $w(x,y;\infty)$ and $f(x,y;\infty)$ remain finite. Integrating the resulting equations and applying the boundary conditions we obtain

$$w(x,y;\infty) = f(x,y;\infty) \equiv 0 \ .$$

To illustrate this result, graphs are shown in Fig's. 9-11 of typical stresses and displacements for Shells I and II on the line $y = \ell$ for an increasing sequence of τ values. All stresses and displacements seem to approach zero as τ increases. The decrease in the magnitudes of the solution as τ increases is another possible cause for the slow rate of convergence for large τ.

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Fig. 7. The variation with x of Σ'_{xy} for Shell I for a sequence of y values.

Fig. 8. The variation with x of the dimensionless normal displacement w for Shell I for a sequence of y values.

Fig. 9. The variation with x of the dimensionless normal displacement w for Shells I and II on the line $y = \ell$ for a sequence of τ values.

Fig. 10. The variation with x of Σ^{o}_{x} for Shells I and II on the line $y = \ell$ for a sequence of τ values.

Fig. 11. The variation with x of Σ'_{y} for Shells I and II on the line $y = \ell$ for a sequence of τ values.

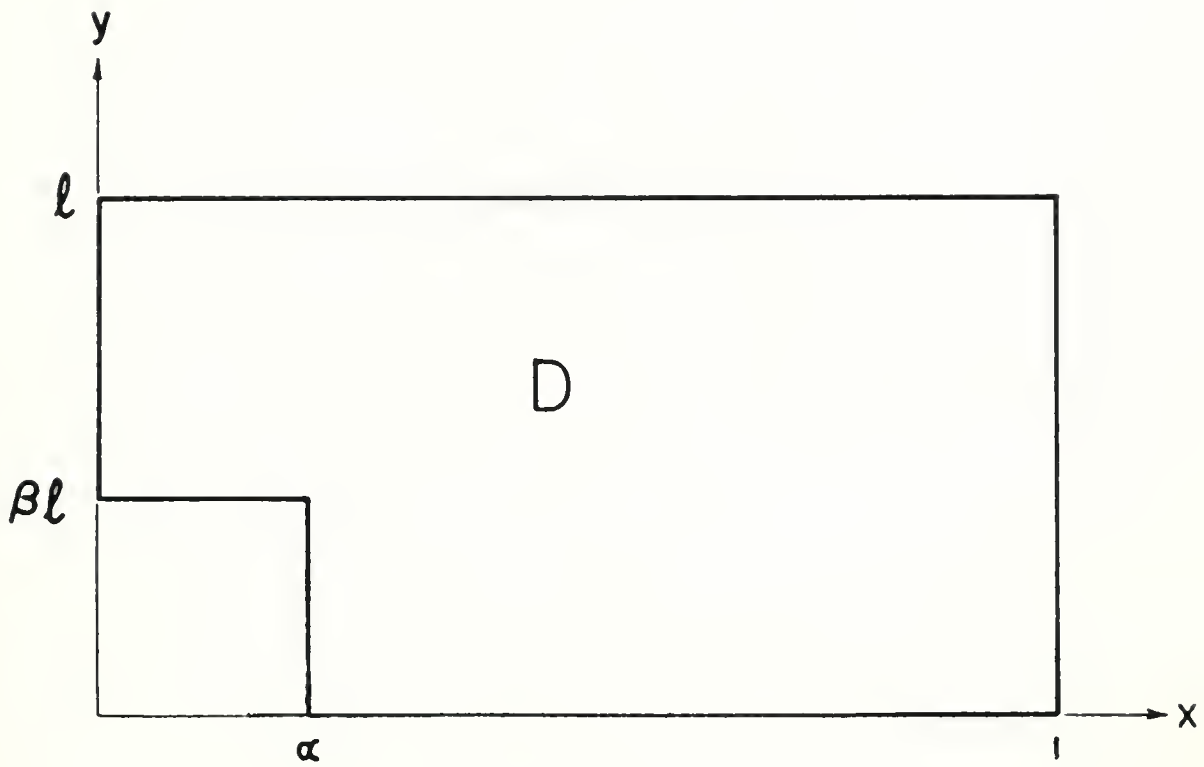
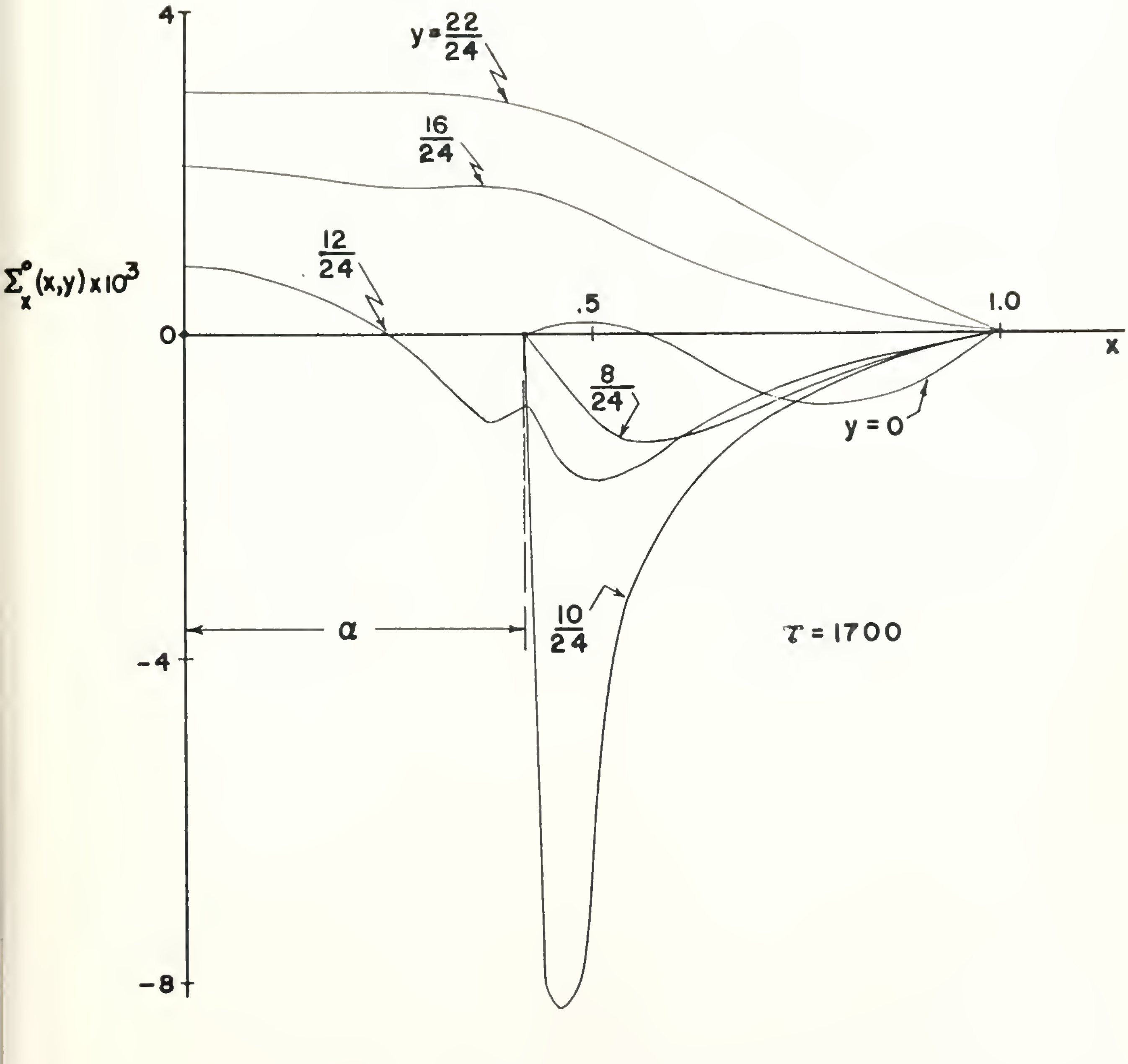


FIG. 1



4
0
-4
-8
$\Sigma_x^\circ(x,y) \times 10^3$
$y = \frac{22}{24}$
$\frac{16}{24}$
$\frac{12}{24}$
$\frac{8}{24}$
$\frac{10}{24}$
$y = 0$
.5
1.0
x
α
$\tau = 1700$

FIG. 2

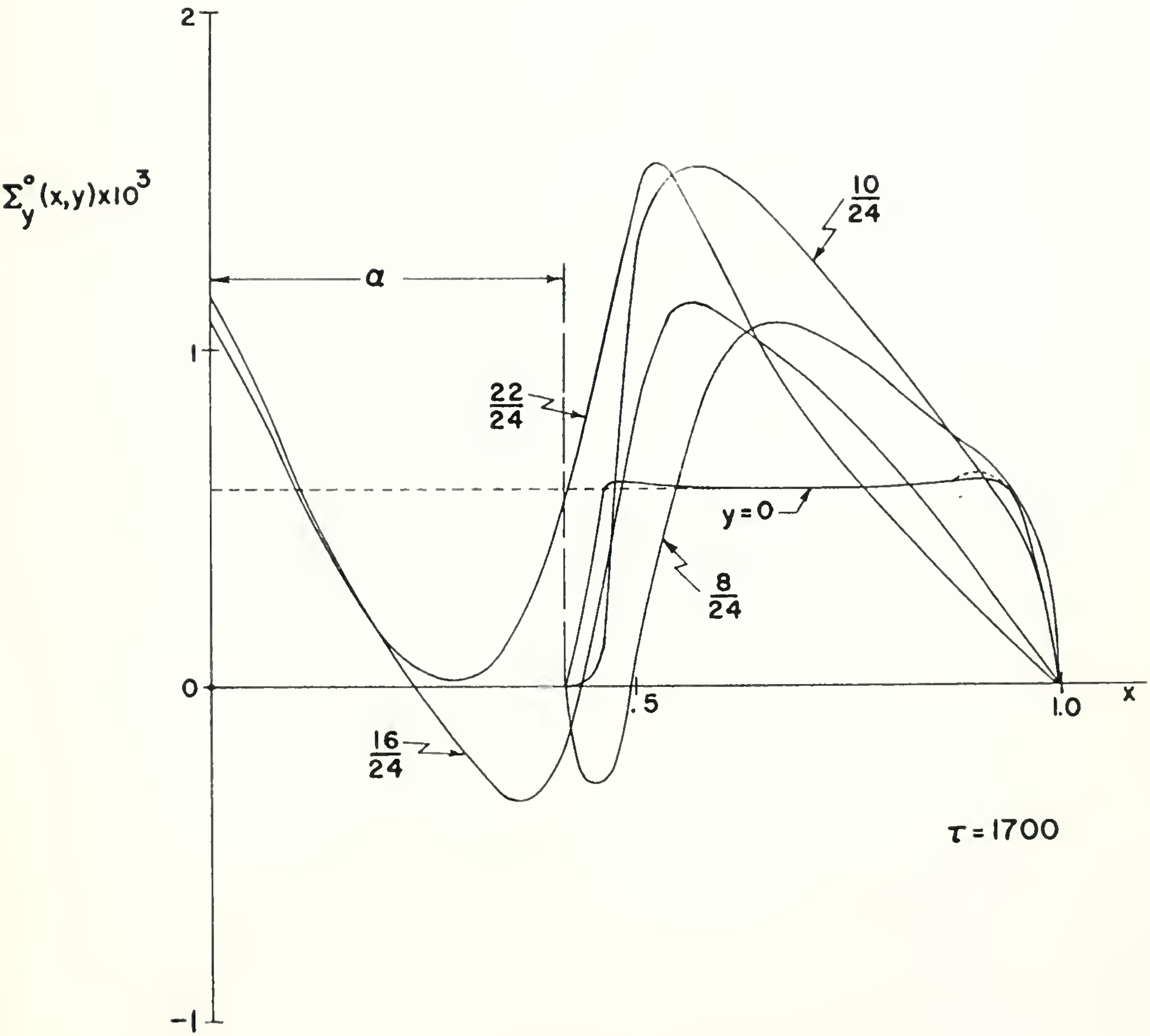

$\Sigma_y^{\circ}(x,y) \times 10^3$
2
1
0
-1
α
$\frac{10}{24}$
$\frac{22}{24}$
$y=0$
$\frac{8}{24}$
$\frac{16}{24}$
.5
1.0
x
$\tau = 1700$

FIG. 3

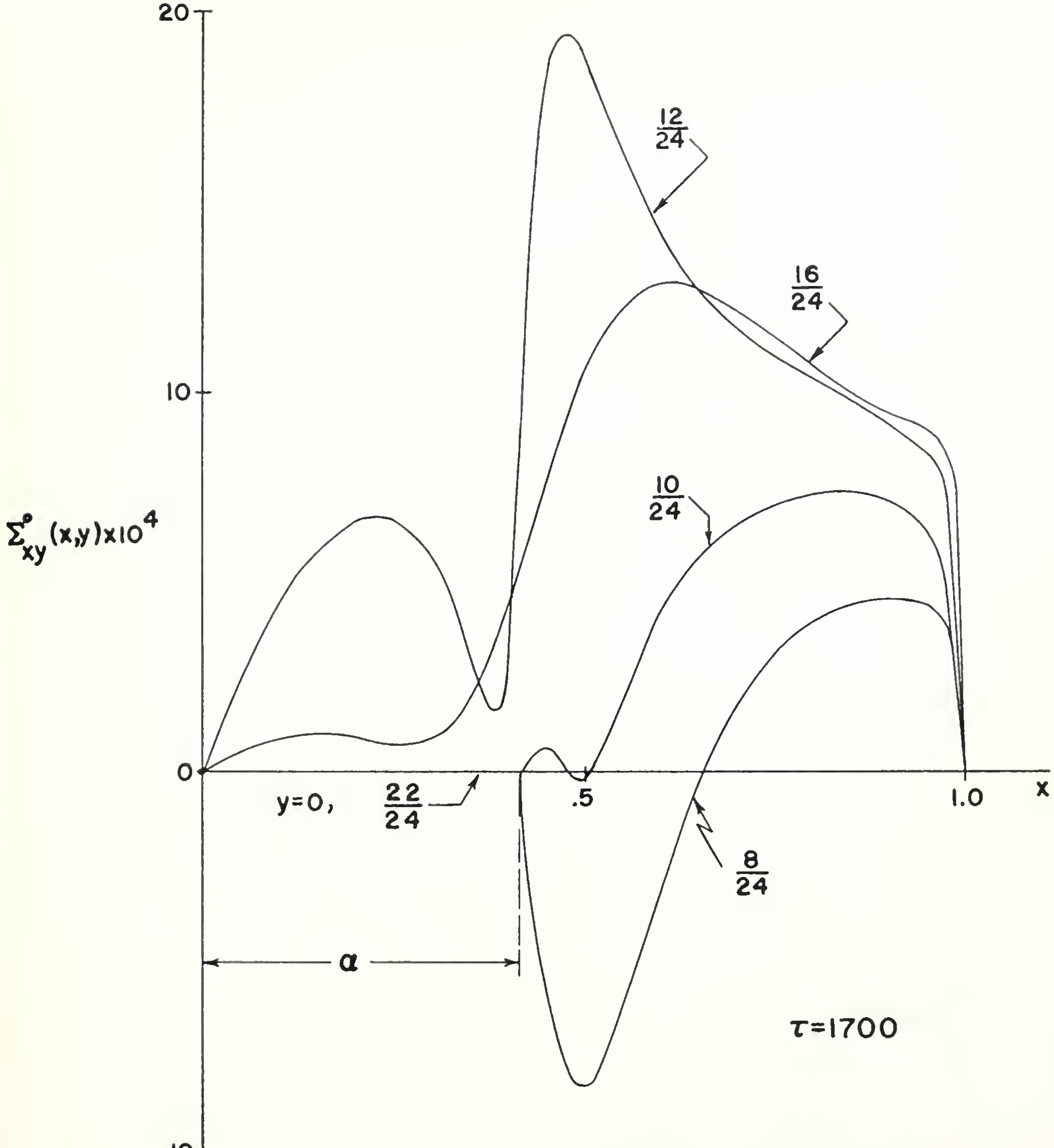
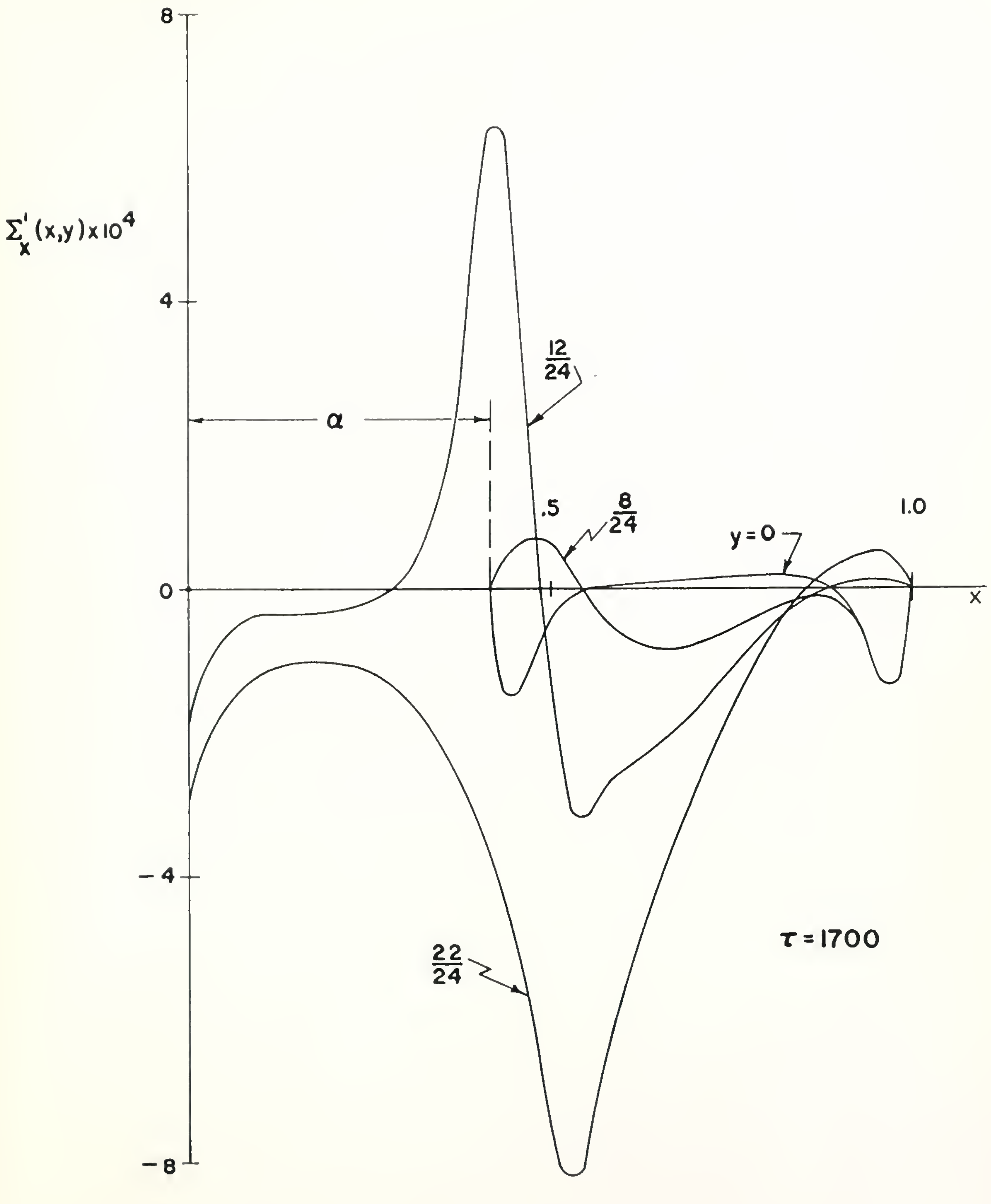

20
10
0
-10
$\Sigma^{\circ}_{xy}(x,y)\times 10^{4}$
$\frac{12}{24}$
$\frac{16}{24}$
$\frac{10}{24}$
$\frac{22}{24}$
$\frac{8}{24}$
y=0,
.5
1.0
x
α
τ=1700

FIG. 4



8
4
0
-4
-8
$\Sigma'_x(x,y) \times 10^4$
$\frac{12}{24}$
α
.5
$\frac{8}{24}$
y = 0
1.0
x
$\tau = 1700$
$\frac{22}{24}$

FIG. 5

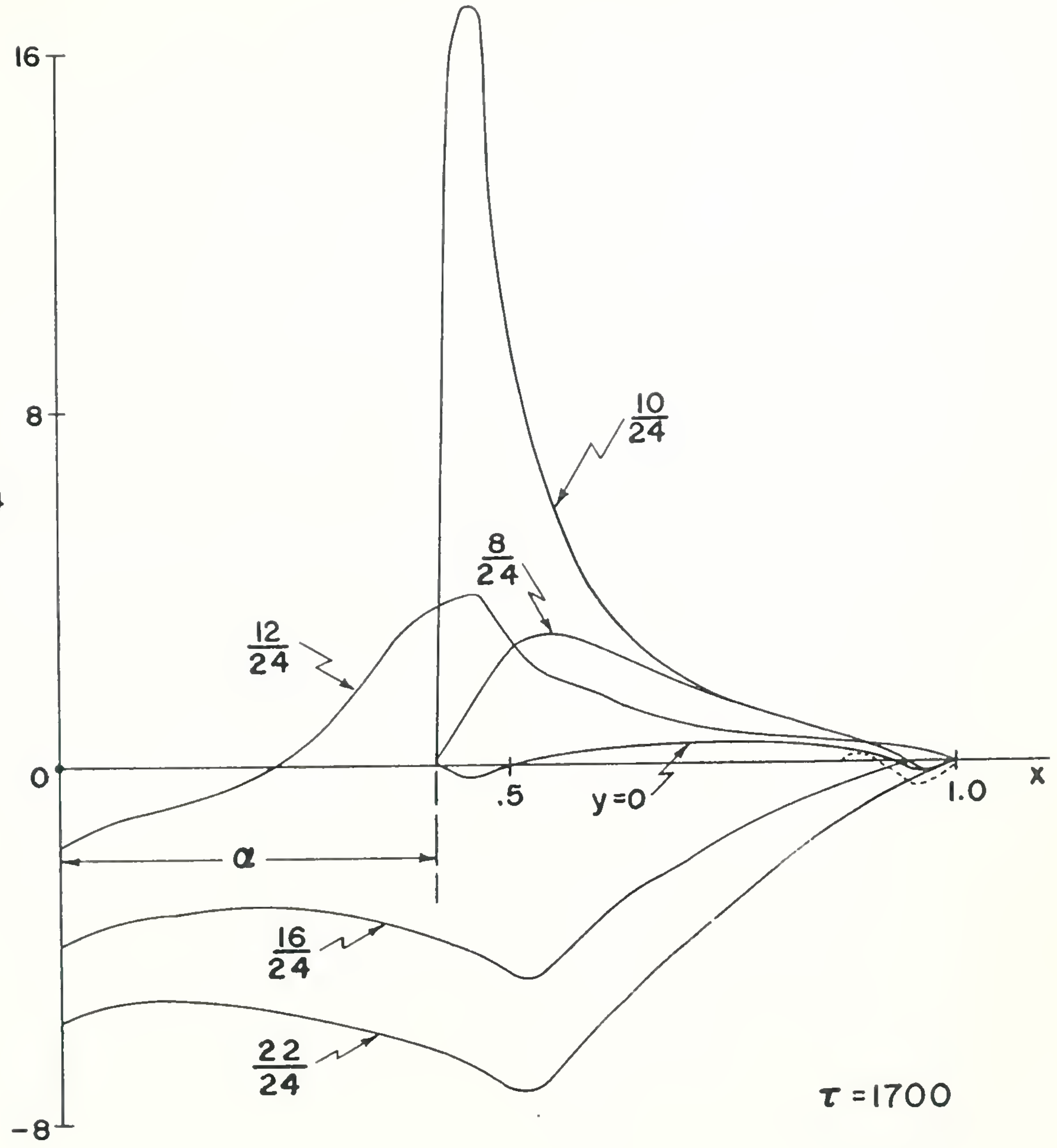

16
8
0
-8
$\Sigma_y'(x,y) \times 10^4$
10/24
8/24
12/24
16/24
22/24
y=0
.5
1.0
x
α
τ =1700

FIG. 6

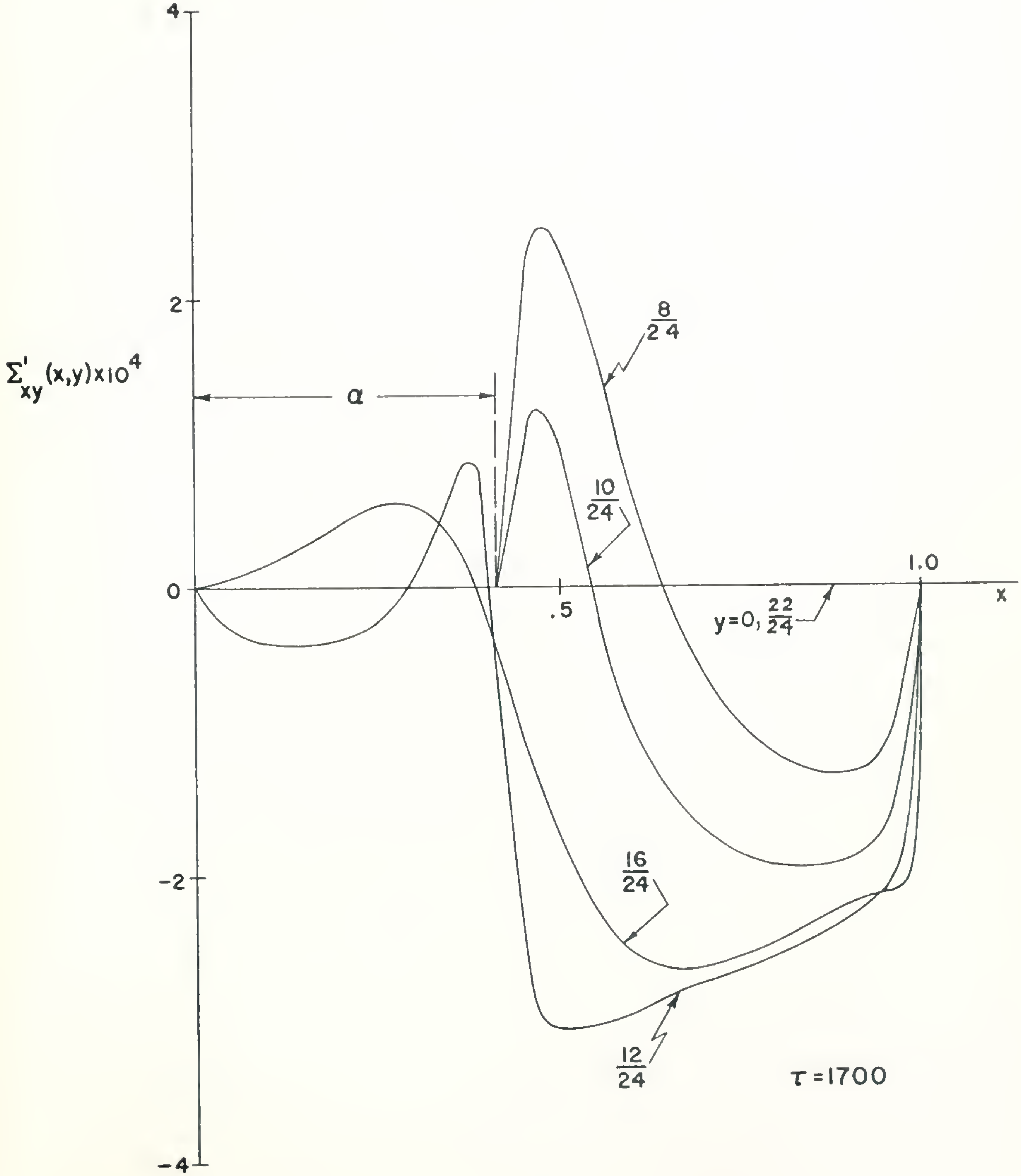

4
2
0
-2
-4
$\Sigma'_{xy}(x,y) \times 10^4$
a
8/24
10/24
16/24
12/24
.5
1.0
x
y=0, 22/24
τ =1700

FIG. 7

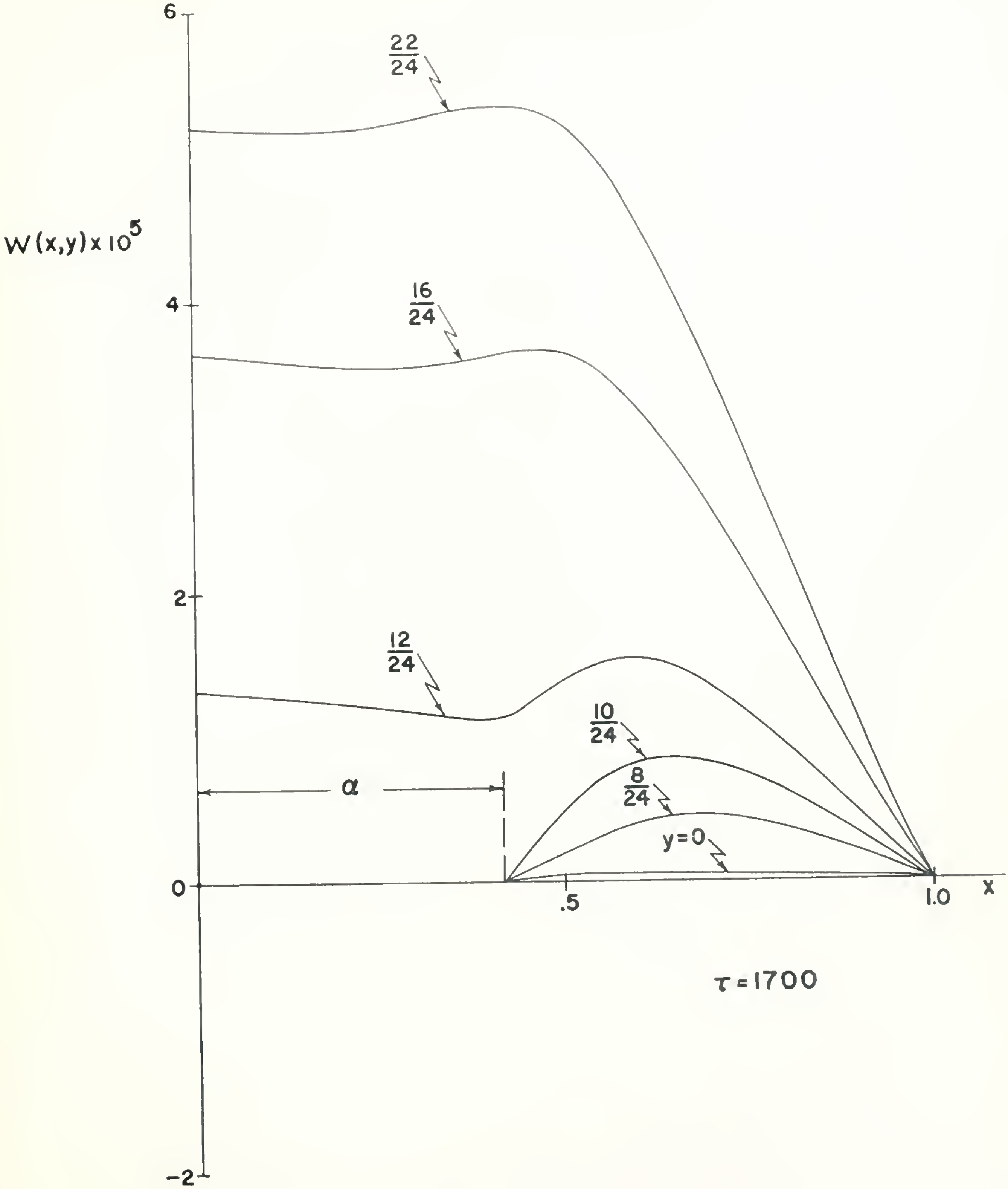


FIG. 8

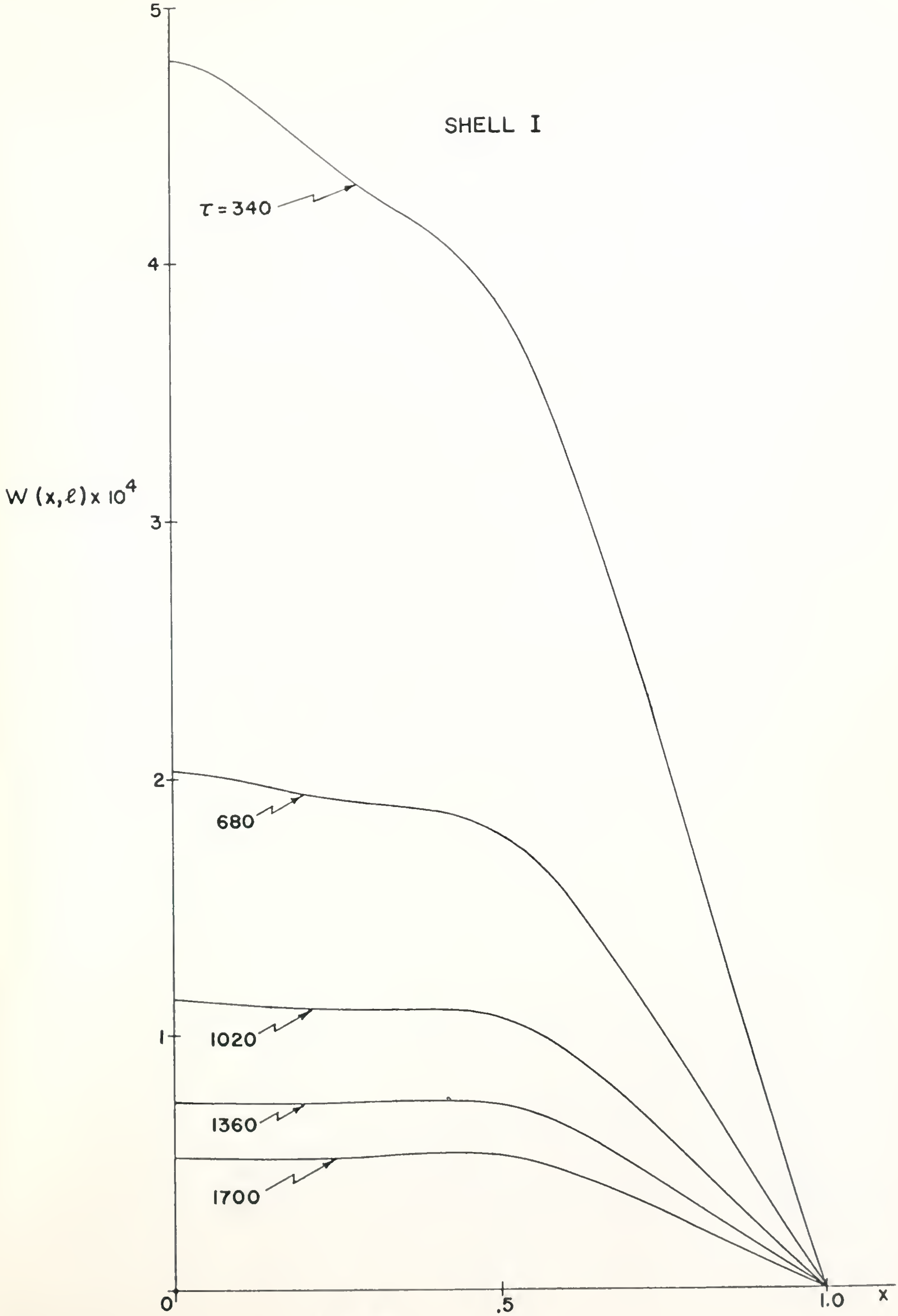


FIG. 9a

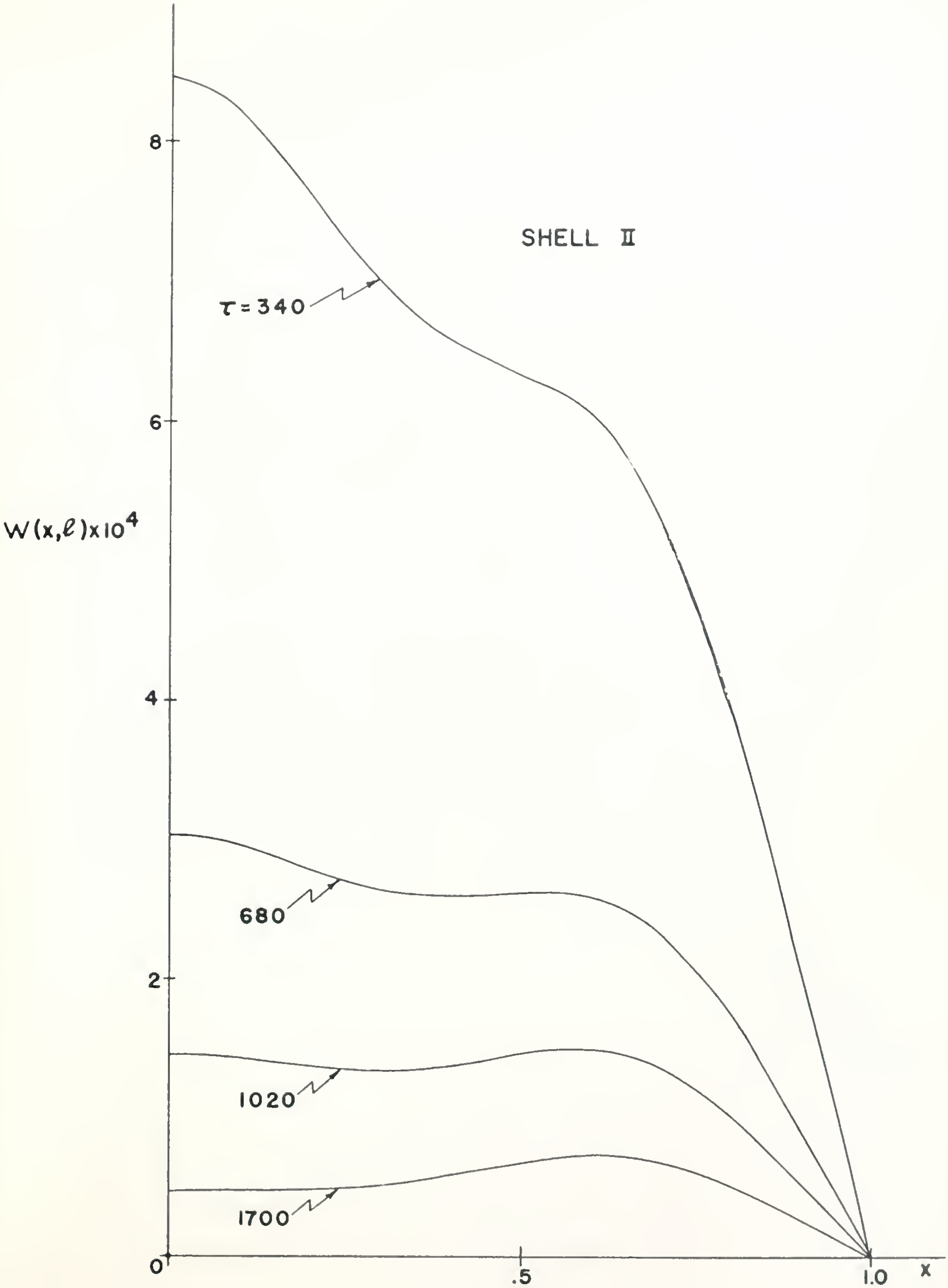
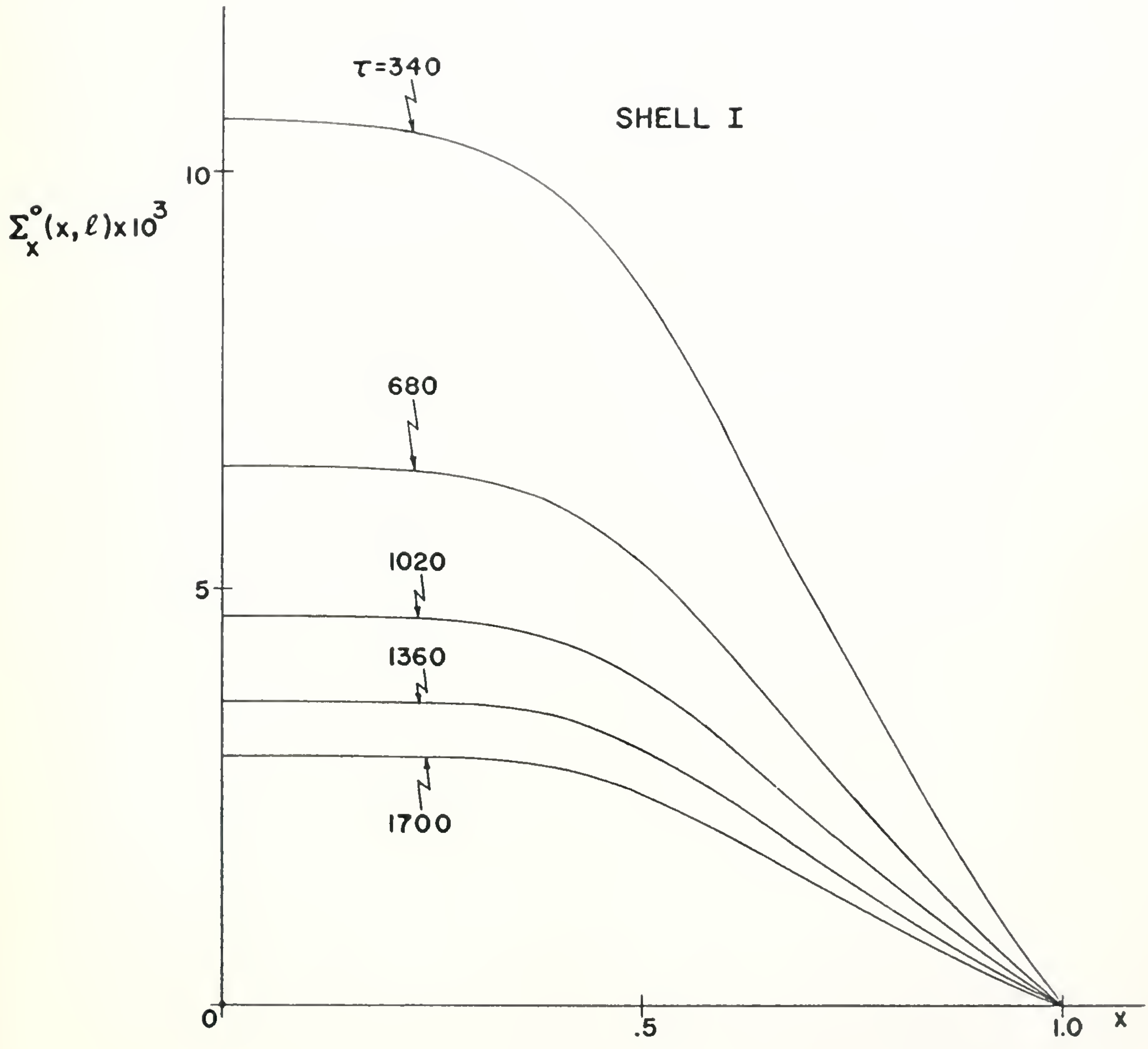
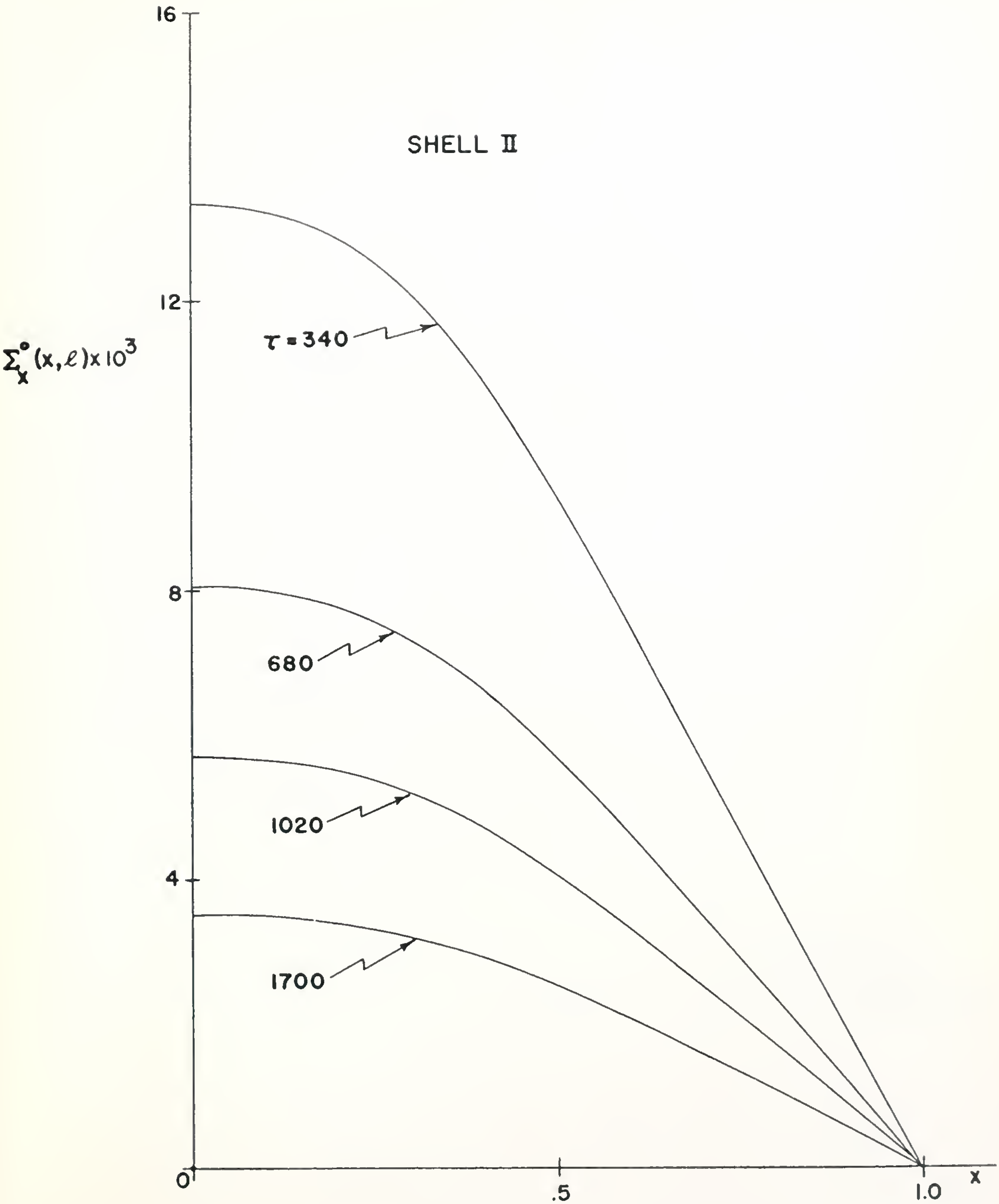


FIG. 9b



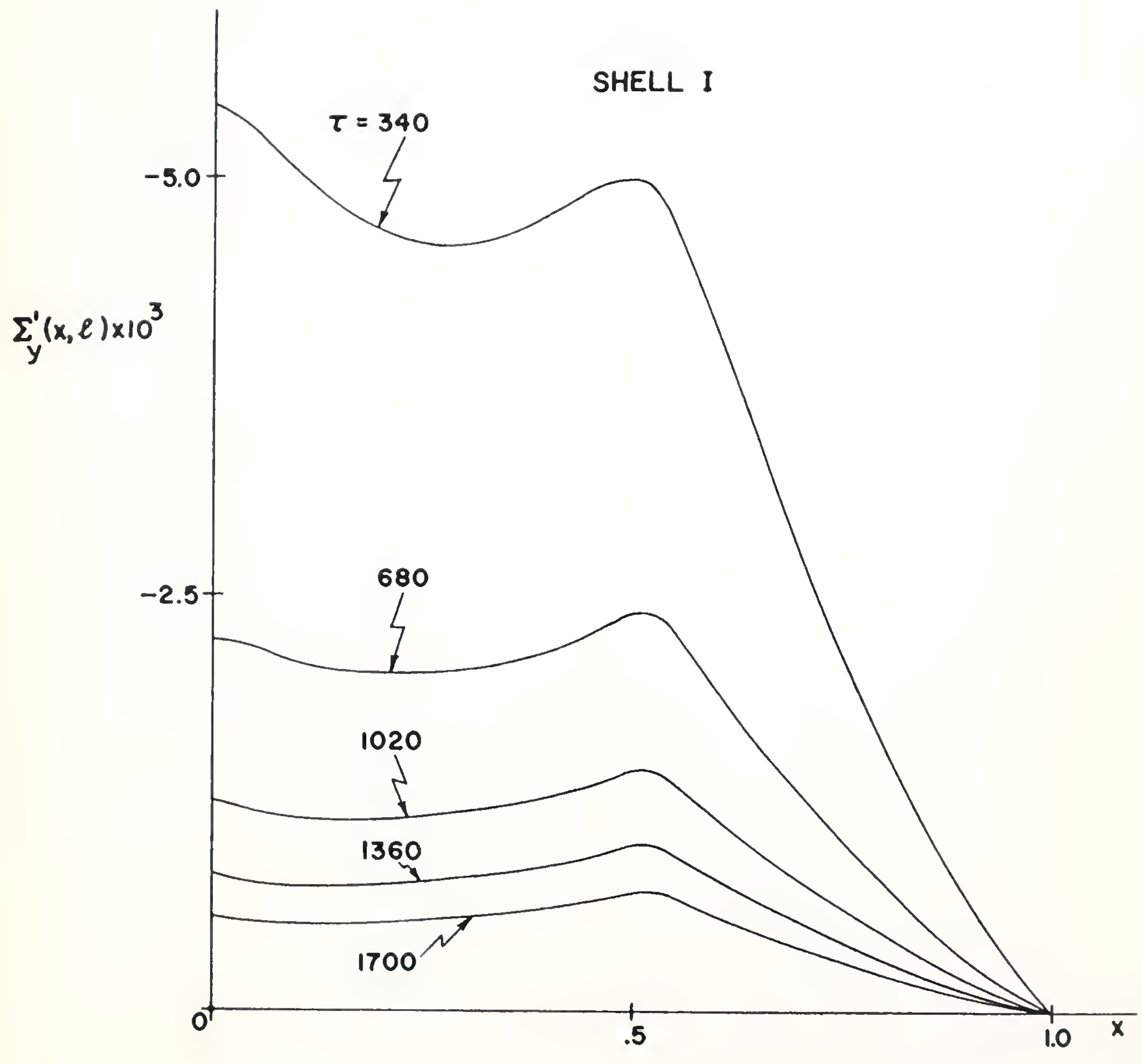
τ=340
SHELL I
10
$\Sigma_x^o(x, \ell) \times 10^3$
680
1020
5
1360
1700
0
.5
1.0
x

FIG. 10a



16
12
8
4
0
$\Sigma_x^\circ(x,\ell)\times 10^3$
SHELL II
$\tau = 340$
680
1020
1700
.5
1.0
x

FIG. 10b



SHELL I
$\tau = 340$
680
1020
1360
1700
-5.0
-2.5
0
.5
1.0
x
$\Sigma_y'(x, \ell) \times 10^3$

FIG. 11a

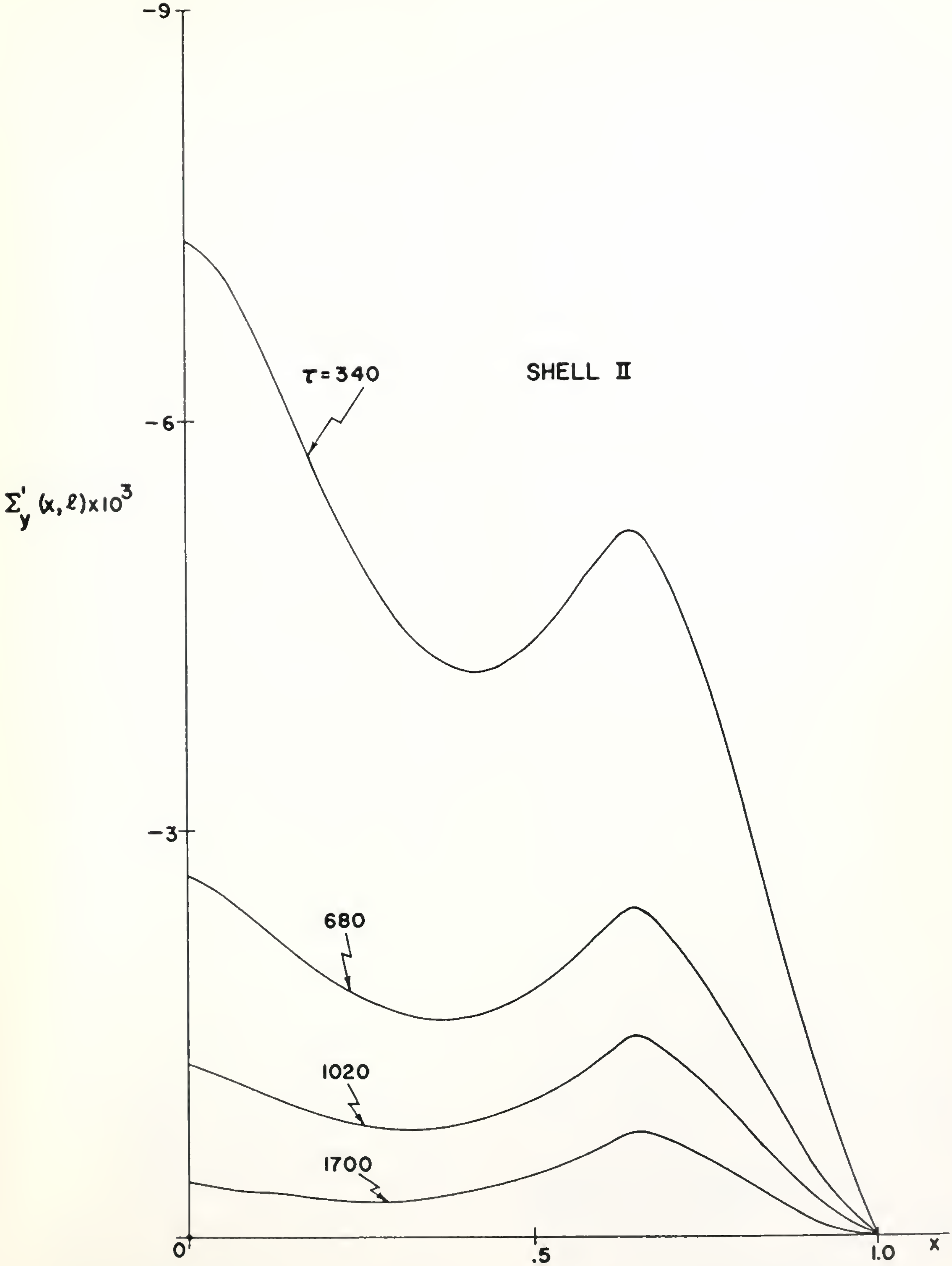

-9
-6
-3
0
$\Sigma'_y(x,\ell)\times 10^3$
$\tau = 340$
SHELL II
680
1020
1700
.5
1.0
x

FIG. 11b

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